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CRITICISMS AND DISCUSSIONS.

THE MEANING OF CHANCE.

Examinations of the concepts of *chance* spread as if by a kind of logical contagion through the three broadly distinguishable fields of philosophy—metaphysics, logic and the theory of knowledge. A confusion of these three aspects of the subject has caused much of the difficulty in determining what a judgment of chance measures, and in showing the place of the laws of chance among the other laws of logic. With the hope of avoiding some of these confusions and of giving a sharper definition to the concept, the present essay will treat chance under these separable heads: the logic of chance, the epistemology of chance, and the metaphysics of chance.

The Metaphysics of Chance.

Chance as a concept often accompanies indeterminism. There can be no real chance in a deterministic world, we are told, nor any determination in a world of real chance. Upon this issue two radically opposed types of metaphysics arise. They are at one only in agreeing that chance is to be linked to some ultimate principle of reality.

Says the determinist: "There is no doubt in the lightning as to the point it shall strike;. . . .not a grain of sand lies upon the beach but infinite knowledge would account for its being there; and the course of every falling leaf is guided by the principles of mechanics which control the motions of the heavenly bodies. Chance can not exist in nature."[1]

If there were a world of real chance, the determinist would look upon it as a chaos. But according as one has or has not a liking for revolution and novelty, one becomes a *tychist* or a strict determinist. Peirce, who invented the term *tychism*, says: "Now the only possible way of accounting for the laws of nature and uniformity in general is to suppose them results of evolution. This supposes them not to be absolute, not to be obeyed precisely. It makes an element of indeterminacy, spontaneity, or absolute chance in nature."[2] Peirce refers to this as "that way of thinking which it will be convenient to christen tychism."[3] Thus, on the one hand, is a Bergson or a James insisting upon change, newness, deviation from law as the ultimate truth about reality; on the other, a Jevons or a Mill holding fast to necessity, and causes and effects.

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[1] Jevons, W. S., *Principles of Science,* Ch. X, pp. 197-8.

A definition of chance which sets out from either of these metaphysical premises must solve the dilemma of chance and necessity by making one or the other a superficial character of experience. Chance must be appearance and necessity reality, or vice versa.

If the universe is tychistic, the causal connections in it will be like habits in the life of a man who has no general plan of living. They are temporary uniformities, crystallizations out of the flux. They will give way to new and different uniformities. No law is eternal except chance; the laws of nature by which events seem to be determined are an insecure crust upon the deeper flow of change.

The usual view is the reverse. Most writers hold that chance is the illusion and necessity, the reality. This view makes a stronger appeal to common sense. Human habits of thought make it easier to believe that "every falling leaf is guided by the principles of mechanics" than to suppose that the principles of mechanics are, themselves, a phase in the lawless transformations of the universe. On this view, every event belongs somewhere in a setting of strict law. If nature appears at times to follow no rule, that is not because there is no rule.

The conventional way of dealing with illusions is to assign them to the mind. This is the classical method of defining chance, with Laplace as authority. Chance is an appearance due to human ignorance. To the all-knowing mind there is only necessity. If we are ignorant of the causes of an event, says this view, we can not predict it with certainty; we can only hold a belief about its occurrence. It seems to follow that the measure of the strength of this belief will be the measure of the chances of the event.

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[2] Peirce, C. S., *The Monist,* Vol. I, p. 164.

[3] *Ibid.,* Vol. II, p. 533.

Having first been an appearance, and then a mental illusion due to ignorance, chance now becomes the degree of intensity of a belief. De Morgan states this position in its most radical form. "By degree of probability we really mean, or ought to mean, degree of belief," he says. "I throw away objective probability altogether and consider the word as meaning the state of the mind with respect to an assertion, a coming event or any other matter upon which absolute knowledge does not exist."[4]

We see what are the results of bringing in the metaphysical question of determinism: chance and necessity are distinguished as appearance and reality, and chance is assigned to the mind. If the metaphysical question be not raised, we can discover a neutral view. Neither chance nor necessity need be fundamental, and at the same time both can be intelligibly accounted for. There will be no metaphysics of chance. The nature of reality, whether changing and lawless or strictly determined, will not be in question.

The sanction of this non-metaphysical view is pragmatic. Experience can be treated both as if it were a series of chance events and as if it were bound together by strict laws. But to treat it in either fashion is not to prove that it is *really* one or the other.

Nature presents all degrees of uniformity from the startling appearance of a new comet to the regular motion of the planets about the sun. To certain uniformities we give the name of natural laws. They are those uniformities which have been tested by many observations and fitted into place with other uniformities, and from which no deviation is known to occur. The laws that light travels in straight lines is such a uniformity. But there are other uniformities (the word being widely interpreted) which hold *on the whole,* though not strictly. Such is the uniformity, extensively verified by experience, that about fifty-one per cent. of all births will be female births. This is a statistical uniformity. It admits of variation and exception. It compromises with the disorderly character of the facts by making its demand for uniformity less rigorous.

Statistical propositions afford a means of handling different degrees of orderliness. So long as experience is sufficiently coherent to present *things* or *events,* the statistical method can make some generalizations about the relations between these things or events. The connections in experience thus generalized will be chance connections. No absolute prediction of any event or thing can be made upon them.

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[4] De Morgan, *Formal Logic,* p. 172.

To say that anything comes about by chance means, then, that it follows from certain conditions with a regularity which is not absolute. The measure of its regularity (or irregularity) will be the chance of its occurrence. No answer is implied as to whether *reality* is regular or irregular, uniform or disorderly, determined or undetermined.

The world seen under the aspect of chance is the world from a different point of view. It is not a different world. If we frame generalizations which hold on the whole, or in a proportion of all the cases which they cover, we are dealing with experience as a thing of chance. If we frame generalizations which hold without exception, we are dealing with it as a thing of necessity.

We may ask for what reason does one wish to view experience as a series of chance events. Why use statistical generalizations?

In the first instance, some statements of fact can be generalized only in the statistical fashion. It is impossible, for example, to state the general conditions of weather changes so strictly that there will be no exceptions. Statistical generalizations are forced upon the meteorologist by the very nature of his facts.

For the Laplacian theorists of chance this would be a case of human ignorance. To the all-knowing mind, they would say, the weather must be an open book. But the all-knowing mind is only another form of the premise of determinism. Completer knowledge would reveal strict connections only if strict connections existed. If there were no rigid uniformities into which the facts would fit, as the indeterministic metaphysics asserts, then no amount of knowledge would produce a rigid generalization. A statistical statement would be the only possible means of generalizing from the facts.

All generalization is a process of extending human knowledge. It is effective in proportion to the range of facts included and to the elimination of human ignorance. Statistical generalizations are no exception; they are an attempt to surmount ignorance. But only upon the deterministic assumption can human ignorance be said to be the *raison d'être* of the statistical method.

Secondly, statistical generalizations may be used for convenience. Strict laws by which the phenomena in question are determined may or may not be known. Such for instance would be the statistical distribution of molecules assumed in the kinetic theory of gases. Each molecule obeys the laws of motion; but it is not convenient to calculate what each molecule would do. Statistics are thus suitable for synoptic views. The all-knowing mind, even in a determined universe, might prefer to class together many events and deal with them as masses, rather than to follow out the tedious details of each. It is obvious that in such cases the extent of our ignorance or knowledge would not affect the chances involved.

Chance, therefore, need not be defined in metaphysical terms. The use of statistics does not imply that reality is either determined or undetermined; and the utility of the statistical method does not rest upon ignorance.

The Epistemology of Chance.

We have shown that it is a short step from the deterministic view of chance to the corollary that a degree of chance is the degree of certainty or uncertainty of a belief.

What is the relation between belief and judgments of chance? This is the epistemological question.

To affirm, for example, that the chances of death at sixty are greater than the chances of death of twenty-five appears to be a plain statement of fact. Like any statement of fact, it may be believed or disbelieved. The intensity of the belief will not constitute its likelihood. If the likelihood of a proposition believed were created by the intensity of the belief, necessity would be created by irresistible conviction. A judgment such as the one just given refers to the facts of experience, as does any other judgment. It states in the last analysis that more people die at sixty than at twenty-five. This is its factual content. If this is true, it will be a correct judgment of chances. We need no special category of knowledge in which to place it.

This is proper enough, it will be answered, so long as we confine ourselves to the statistical generalization—that the proportion of deaths among men of sixty is greater than the proportion of deaths among men at twenty-five. But suppose we inquire about the probabilities of Tom Jones's death at these two ages. We must then affirm that a particular proposition, "Tom Jones will die at twenty-five," has a probability. Is this a statement of fact? Certainly we do not affirm the fact to which the particular proposition refers: we do not say that Jones will die at twenty-five. Neither do we deny it. But we do affirm something, which is a fact, about the whole proposition. We say it has a certain probability. Now, the probability that Jones will die at twenty-five is the same as the probability that *any* man will die at twenty-five. Jones is a specific value for the indefinite article "any." What we are affirming about the probability of this particular proposition can also be affirmed about a class of propositions, viz., about all the propositions which assert that someone will die at twenty-five. It is because the particular proposition is one of a class of such propositions that the particular proposition can be judged probable. The judgment of probability is a statement of the fact that a certain number of the propositions in this class are true and a certain number false. The statistical generalization that "*ten per cent.* of all men die at the age of twenty-five" is another way of stating that "*ten per cent.* of all propositions which assert that some man will die at twenty-five are true."

The relation between a judgment of the chances of a particular event and the statistical generalization upon which it is based is a matter of logic, and will be treated as such under the logic of chance. It rests upon the axiom that what is true of all members of a class is true of a particular member. Statistical generalizations deal with classes of propositions; and hence, the frequency of truth found in the class will be the probability of any particular proposition of the class. One may either correctly or erroneously believe that this truth frequency has a certain value. But the belief will not affect the value.

This is simple enough and yet the point has been a source of endless confusion. A particular proposition, we are told, must be either true or false, with no middle ground. Probability seems to be such a middle ground and, therefore, it must be an illusion. But it is evident that a particular proposition may be true or false and also probable. "Tom Jones dies at twenty-five" will be true or false as the facts decree. It may be also probable, because the judgment of its probability does not either affirm or deny that the proposition is true or false. There is no contradiction here. The judgment of its probability lies in another plane.

Bound up with the view that chances are created by human beliefs is the prejudice that chance refers essentially to the future. But whether the event whose likelihood is in question has or has not come to pass, it remains a chance event. Either before or after Tom Jones's death, the probability that he will die before he is twenty-five has a meaning and a numerical value. This is a corollary of what we have said before. There is no absurdity in speaking, after Jones's death, of the probability that he might not have died, which is an admission that his death is still in the realm of the probable, though it has actually come to pass. Very often past events are spoken of as improbable, surprising, or unlikely. If probability did not exist after the event as well as before, such a habit of speech would be pure nonsense.

It must be concluded that chance is as objective as any matter of fact. It belongs in no way peculiarly to the belief side of knowledge. It is a matter of experience to be handled by observation and ordinary methods of verification.

The Laplacian theory of chance is often called objective, but it is far from being objective in the sense just defined. Laplace says: "The probability of an event is the ratio of the number of cases which favor it to the number of all possible cases, when nothing leads us to believe that one of these cases ought to occur rather than the others, *which renders them for us equally probable*."[5] It is the last (italicized) clause of this definition which makes probability objective. This is the equal distribution of ignorance. The fundamental axiom of the theory is: where nothing is known about the likelihood of an event, its probability is one to two. It is as likely to happen as not.

This axiom may be taken as self evident or it may be grounded in experience, as it is by Edgworth. "I submit, the assumption that any probability constant about which we know nothing in particular is as likely to have one value as another, is grounded on the rough but solid experience that such constants do as a matter of fact as often have one value as another."[6]

In any case, ignorance is a very poor ground upon which to base knowledge. We have mentioned reasons for disregarding ignorance as a factor in chance. It is peculiarly difficult to understand why we should be permitted to prescribe that an event of whose conditions we know nothing must occur in fifty per cent. of all cases. Any other guess would be equally permissible. Each *a priori* guess of the probability of an event must wait for the facts to corroborate or refute it. The statistical method must accept the irregularities of experience as it finds them. There can be no equal distribution of ignorance.

Judgments of chance, therefore, show no epistemological peculiarity. Judgments of chance differ only in one respect from particular judgments of fact. They are judgments *about* propositions. Their subject matter is the relation of a certain proposition to certain other propositions. This is still within the realm of fact.

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[5] Laplace, *Theorie Analytique des Probabilites,* liv. II, Ch. I, no. 1.

[6] Edgworth, *The Philosophy of Chance,* Mind, N. S., Vol. IX (1884), p. 230.

We may now ask specifically what is the meaning of the statement "that a proposition, p, has a probability, *P*." This will lead to the logic of chance.

The Logic of Chance.

There are two distinguishable varieties of chance: restricted chance and unrestricted chance. Another name for unrestricted chance is bare or antecedent probability.

On the Laplacian theory, bare probability is associated with complete ignorance. The philosopher who says that *a priori* all things are equally probable (or possible) has in mind unrestricted chance.

We may ask if a bare or unrestricted probability is not a contradiction in terms.

We have said that so long as experience is sufficiently coherent to present *things* or *events,* the statistical method can be applied. This is a condition, very general, to be sure, which experience must fulfill in order that chance may have any meaning. Before we can even speak of the antecedent probability of a thing or event, experience must give us things and events. Peirce says, on this point, that the application of the statistical method demands a universe with some character. "The presupposition of the determinate constitution of any set of facts such as are subject to inductive investigations is by no means a simple, not even a 'self-evident,' presupposition....But the presupposition, as Peirce has shown, is the one natural and indispensable presupposition in all inductive science."[7]

It will be convenient to substitute for the terms *things* and *events,* the more general term, *propositions.* Propositions assert the existence of things or the occurrence of events.

The minimum condition, then, upon which any proposition can be barely probable is that it shall be at least a proposition, capable of truth or falsity. This is the meaning of Peirce's statement that chance demands a universe with a constitution.

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[7] Royce, J., *Encyclopædia of the Philosophical Sciences,* edited by A. Ruge, Vol. I. p. 83.

We are here in contact with something equivalent to the logical universe of discourse. The widest context in which chance has a meaning is the world of any or all events, or the world of any or all propositions. Unrestricted chance is, therefore, chance under no special condition, but under the most general possible condition.

Specific statistical researches do not deal with antecedent probabilities. Statistics are always gathered within a limited field, i. e., a restricted universe of discourse. They are significant only upon the special conditions which hold within that universe. The *consistency* of a set of statistical propositions demands a strict adherence to the limiting conditions.

When the probability of a proposition depends, thus, upon the truth or falsity of some other specific proposition, we have a case of restricted chance. We shall call it "probability upon." It is in this form that probability is most familiar and useful. The probability of bad weather, for instance, is not a bare probability. It is not probability *in vacuo* under the widest possible conditions. It is based upon specific conditions: a certain degree of humidity, a certain velocity of wind, a certain geographical situation. Given other specific conditions, the probability will differ.

The difference between restricted and unrestricted probability may be summed up as follows: If a proposition, p, is probable upon a proposition, q, the latter is recognized as a special condition which must be considered before the probability of the former properly has a value other than its antecedent probability. On the other hand, the antecedent probability of the proposition, p, will depend upon no special condition. It will be the probability of p upon the most general conceivable condition, viz., simply that p is a proposition, true or false.

It is clear that the proposition which is probable upon another proposition stands in some sort of logical relation to it. This relation can not be the relation of formal implication, since any implied proposition can be strictly inferred from the proposition which implies it. A probable proposition can never be strictly inferred or predicted from its conditions. The relation is, however, like formal implication. It is a kind of *informal* implication. It takes the "if—then" form which is characteristic of implications.—"*If* certain atmospheric conditions are present, then probably bad weather will result."

The relation may be called *contingency*. The degree to which one proposition or set of propositions is contingent upon another proposition or set of propositions is, in the first instance, the subject matter of judgments of probability.

It will be necessary to analyze the relation of contingency.

In logic the term proposition usually applies to a particular statement which is capable of truth or falsehood. We shall mean by a proposition what Mr. Russell calls an elementary proposition—"a proposition such as 'this is red' where 'this' is something given in sensation."[8] Thus, a proposition is true or false at some particular time and in some particular situation. But a proposition can be generalized so that what the proposition asserts is true or false at many different times and in many situations. If in place of "this is red" we assert "x is red," we obtain a whole series of elementary propositions by giving different values to the x.

Contingency is a relation between generalized propositions in which an undesignated or variable term appears. Consider the judgment, "If I am descended from a long-lived family, I shall probably be long-lived." The "I" is merely a particular value of an x; and the judgment remains the same when "anybody" or x is substituted for me.

There is a frequency with which the ambiguous proposition "x is descended from a long-lived family," will coincide in truth with "x is long-lived." This is the degree of contingency of the latter upon the former. The single coincidence of two elementary propositions is uninteresting and unimportant for the theory of chance. Such a coincidence would not give a degree of contingency. Degrees of contingency arise where there is a series of coincidences or failures of coincidence.

The generalized, or ambiguous proposition of which we spoke is a propositional function.[9] So that contingency is a relation between propositional functions.

The propositional function stands for a collection of propositions none of which is asserted. For the purpose of formal logic there are two ways of asserting a propositional function: (1) we may assert "ϕx always"; i. e., all propositions which are values of the function. This will be formal or universal assertion. (2) We may assert "ϕx sometimes"; which is tantamount to saying "there exists a value for which the function is true." This will be a particular assertion.

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[8] Whitehead and Russell, *Principia Mathematica,* Vol. I, pp. 95-96.

[9] Whitehead and Russell, *op. cit.,* p. 15.

Statistics deal with assertions which lie between the universal and the particular, and which are, in De Morgan's phrase, "numerically definite." For the purposes of statistics it must be possible to assert, "ϕx in n cases"; or "there exist n values for which the function is a true proposition." By means of this concept—"the n true propositions which are values of the function ϕx," where this totality is finite, probability and other statistical ideas can be given a strictly propositional interpretation.

The degree to which one propositional function is contigent upon another can be stated in numerically definite terms as follows: There exist n true values of a certain function, ϕx. There exist, also, m true values of the coincidence of this function with another function, ψx. The proportion m/n will measure the degree of contingency of ψx upon ϕx. (The conjunction or coincidence "ϕx and ψx" is itself a propositional function which, like any function, may be true for all values, for some values, or for m values.)[10]

This is exactly what we mean when we judge, "the probability is P that if x is descended from a long-lived family he will be long-lived." There are a number of cases in which it is true "that x is descended from long-lived family *and* x is long-lived." The proportion of these cases to all the cases in which it is true "that x is descended from a long-lived family" will be the degree of contingency of longevity upon descent from long-lived ancestors.

In general, then, the degree of contingency of one propositional function upon another will be determined by two existential propositions: (1) There exist m true values of the conjunction of the functions, and (2) there exist n true values of the conditioning function. We may look upon this "degree of contingency" as a relation, just as disjunction or implication is a relation. But we are dealing in this instance with more than a single relation. We have a class of relations. Wherever a ratio such as that just defined exists, the functions involved stand in a contingency relation. But for different cases we shall have different contingency relations, different degrees of contingency.

The contingency relation will be a formal relation. It will hold for all values of the variable. Thus, in the example given above, whatever value x assumes, the relation will still be the same between longevity and descent from a long-lived family.

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[10] The case in which either n or m is infinite is not included in the definition. It is essential that they be finite numbers, since a ratio with an infinite term has no numerical interpretation applicable to the concept of probability.

It is because contingency is a formal relation that any two elementary propositions which are values of contingent propositional functions can be said to be contingent. "What holds of *all* holds of any." This is a primitive proposition of logic. So, if "John Smith" be substituted for x in the example just given, it will be true that the particular proposition "John Smith will be long-lived" will be contingent in the degree P upon the particular condition "John Smith is descended from a long-lived family." We have said above that such a judgment does not commit us to stating the truth or falsity of either of these two elementary propositions. They may both be utterly false. John Smith may be a saw-dust doll, who has neither ancestors or life. That will not invalidate the judgment of probability, for the judgment says that "*if* John Smith were descended from a long-lived family he would be long-lived in all probability."

It is now clear what is the subject matter of judgments of chance. They affirm that a certain formal relation of contingency subsists between propositional functions. They are to be classed with the hypothetical judgments of traditional logic, which assert the subsistence of the relation of formal implication.

How does the formal relation of contingency appear in cases of unrestricted or antecedent probability? There the probability depends only on the most general condition: that the proposition in question *is* a proposition, capable of truth and falsity. It depends upon the general state of the universe. The antecedent probability of a proposition will be the ratio of all true values of the propositional function, of which the proposition in question is a value, to all values of the function, both true and false. Thus, the antecedent probability of "x will die" will be the proportion of all cases in which this is true to all cases in which it could be either true or false. This will be its degree of contingency upon its mere subsistence as a proposition in the widest possible universe of discourse. Antecedent probability, can, therefore, be defined in terms of restricted probability or "probability upon." It is probability upon the unrestricted universe of discourse.

Much complexity may appear in the chance relations of propositions. So far we have spoken only of the contingency of one proposition upon another. The conditions upon which a proposition is contingent may be many. Similarly, the contingent proposition may be, not a single proposition, but a complex system of propositions. As a simple illustration, consider life insurance. The conditions upon which life is dependent are not health alone, but also age, occupation, residence, etc., and each of these has a different weight. A weather forecast usually states a system of probable events, as "changing winds from the north-west or north, light clouds and a falling temperature."

Judgments of chance deal, for the most part, with contingency relations between systems of propositions. These systems are complex logical combinations of propositions. The laws of all possible combinations are the laws of chance.

There are three important phases of the relation of contingency of which we have not yet spoken: (1) favorable contingency, (2) unfavorable contingency, and (3) independence.

The following is a general illustration of these three concepts. The chances that I will be killed in war are, let us say, two to one. If my duties keep me in an *abri* thirty feet underground, this will increase the probability of my survival and decrease the probability of my death. It will be favorable to the one and unfavorable to the other. Suppose that there are no *abris* thirty feet deep, that the *abris* are as unsafe as any other place of duty. In this case the probability of my survival will be independent of whether my duty places me in an *abri*.

The distinction between antecedent and restricted probabilities must be considered in this connection. The favorable or unfavorable effect of additional conditions will always be relative to the original conditions of a probability. A condition which is favorable under one set of original circumstances might be unfavorable under another set. Independence is similarly affected by the original conditions. Thus, the probability that I would vote the Republican ticket, if I lived in Massachusetts, might be increased if the Democrats threatened the woolen industry by lowering the tariff. On a different original condition, say that I live in Georgia, the attitude of the Democratic party on the tariff might have the opposite effect on the probability of my voting the Republican ticket.

In cases of restricted probability, additional conditions are therefore *relatively* favorable to, unfavorable to, or independent of, the probability in question.

In the case of unrestricted probability, an additional condition which altered the probability could be said to be *absolutely* favorable or unfavorable. An additional condition which left the probability the same would be *absolutely* independent. Thus, for Leibniz, all worlds were *a priori* equally probable. But the goodness of this world was a condition which was absolutely favorable to it, because it increased its antecedent probability.

Summary.

The results of our examination of the concept of chance may be summarized as follows:

The definition of chance does not involve the metaphysical question of determinism. Both chance and necessity appear in the world of experience. Necessity appears when we make generalizations which hold without exception; chance appears when we make generalizations which hold in a proportion of the cases which they cover, i. e., statistical generalizations.

The statistical method is useful (1) because certain facts are such that no strict generalizations (i. e., generalizations without exceptions) can be made about them. (2) The statistical method has utility, also, as a means of dealing with large masses of things and events of which we wish to know the average or general behavior. The utility of the statistical method does not depend on human ignorance.

Judgments of chance are judgments of fact. Their subject matter is a frequency of truth within a class of propositions. Chance is therefore objective and is not created by belief. The equal distribution of ignorance, which is the basis of the Laplacian theory, is inadmissible.

Chance shows two major varieties: (1) unrestricted, and (2) restricted. Restricted chance, or probability upon certain conditions, is fundamental, and in terms of it the unrestricted or antecedent probability can be defined.

Restricted probability, or "probability upon," is a character which propositions acquire by being in the contingency relation to some other proposition or set of propositions. The contingency relation is a formal relation which subsists between propositional functions. Hence judgments of chance are like the traditional hypothetical judgments of logic: they assert a formal relation. The laws of chance show all the possible ways in which this contingency relation may be combined with the other logical relations.

It remains to set down in systematic form some of the most important propositions of the logic of chance.

Symbols:

The symbols p, q, r, s, etc., will stand for propositions.

By 'p,' 'q,' 'r,' 's,' etc., we shall mean all true values of some propositional function ϕx, ψx, etc., of which p, q, r, s, etc., are values. This must always be a finite number. The symbol 'p' can be read eliptically "all cases of p true." (When it is written '$p\ .\ q$,' or 'p .v. q' it means all true cases of the whole expression included in the single quotation marks.)

The contingency relation will be symbolized by the mark, /. So that p/q will mean the probability of p upon q.

Other logical relations will be symbolized by the conventional signs. Disjunction by, v; conjunction by a dot, .; negation by, $\backsim$.

Definition of probability.

The probability of p upon q is the ratio of all true cases of p and q to all true cases of q.

$$p/q = \frac{`p.q\text{'}}{`q\text{'}}$$

Definition of relative independence.

When the probability of p upon q is equal to the probability of p upon q and r, then p is independent of r with respect to q.

$p/q = p/q\ .\ r$ means p is independent of r with respect to the condition q.

The converse is true: that if p is independent of r with respect to q, r is independent of p with respect to q. The relation of independence is symmetrical. (Proved by means of the definition of probability.)

Relative favorable and unfavorable dependence.

If the probability of p upon q and r is greater than the probability of p upon q, then p is favorably contingent upon r with respect to q.

$p/q\ .\ r > p/q$ means p is favorably contingent on r with respect to q.

If the probability of p upon q and r is less than the probability of p upon q, then p is unfavorably contingent upon r with respect to q.

$p/q \,.\, r < p/q$ means p is unfavorably contingent upon r with respect to q.

Definition of antecedent probability.

The *antecedent probability* of p is its probability upon the condition, p, or not-p.

$p/p \;.\mathrm{v}.\, \neg p$. is the antecedent probability of p.

(The condition p or not-p completely exhausts the universe of discourse; it is the widest possible condition upon which p can be probable.)

Absolute independence, and absolutely favorable and unfavorable dependence.

If the probability of p upon q is equal to the antecedent probability of p, P is absolutely independent of q. In this case, $p/p \;.\mathrm{v}.\neg p = p/q$.

If the probability of p upon q is greater than the antecedent probability of p, then p is absolutely favorably dependent upon q. In this case, $p/q > p/p \;.\mathrm{v}.\neg p$. If $p/q < p/p \;.\mathrm{v}.\sim p$, then p is absolutely unfavorably dependent upon q.

Multiplication of probabilities:

The product of two independent probabilities is the probability of the joint contingency of the two propositions upon their joint conditions. This can be stated more specifically as follows:

If p/q is independent of $r \,.\, s$, and r/s is independent of q, then $p/q \times r/s = p \,.\, r/q \,.\, s$.

Proof:

Since p/q is independent of $r \,.\, s$, and r/s is independent of q, therefore $p/q = p/q.r.s$, and $r/s = r/s \,.\, q$. (By definition of independence.)

Therefore: $$p/q \times r/s = p/q.r.s \times r/s \,.\, q$$

$$= \frac{‘p.q.r.s’}{‘q.r.s’} \times \frac{‘r.s.q’}{‘s.q’} \qquad \text{(By definition (of probability.}$$

$$= \frac{‘p.q.r.s’}{‘s.q’} \qquad \text{(By cancellation.}$$

$$= p \,.\, r/s \,.\, q \qquad \text{(By definition of probability.}$$

This is the most general rule for the multiplication of probabilities. Its more familiar and special form is:

The product of the probabilities of two independent propositions is the probability of their joint truth:

$p/q \times r/q = p\ .\ r/q$ where p is independent of r with respect to q.

Addition of probabilities:

The sum of the probabilities of two exclusive propositions is the probability that either one or the other is true.

If the propositional functions, of which p and q are values, are mutually exclusive, then $p/q + r/q = p\ .\mathrm{v}.\ r/q$.

The proof of this proposition requires an assumption which we may call the *principle of addition*. This assumption is: If ϕx and ψx, of which p and q are values, are mutually exclusive, then $`p\ .\mathrm{v}.\ q' = `p' \times `q'$; which means that all cases of p or q true are equivalent to all cases of p true plus all cases of q true.

The proof of the rule for adding probabilities then becomes:

$$p/q + r/q = \frac{`p\ .\ q'}{`q'} + \frac{`r\ .\ q'}{`q'} \qquad \text{(By definition (of probability.}$$

$$= \frac{`p\ .\ q' + `r\ .\ q'}{`q'} = \frac{`p.q\ .\mathrm{v}.\ r.q'}{`q'} \qquad \text{(By principle of (addition.}$$

$$= \frac{`q\ (p\ .\mathrm{v}.\ r)'}{`q'} \qquad \text{(By the distributive law of formal logic.}$$

$$= p\ .\mathrm{v}.\ r/q \qquad \text{(By definition of probability.}$$

The rules for the addition and multiplication of probabilities are the two most important laws of chance. Multiplication, as shown above, depends upon the definition of probability and the definition of independence. Addition requires a postulate: that the sum of the true values of two exclusive propositional functions is equal to all true values of their disjunction. Addition and multiplication, together with the definitions of independence and antecedent probability, give the material for a complete logic of chance.

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